

Recent Developments in Supersymmetric Quantum Mechanics*

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Abstract

Some recent results in symmetric quantum mechanics are presented. A new semi-classical approximation formula for Witten's realization of symmetric quantum mechanics is discussed. Analytical as well as numerical results obtained from these formulas are compared with the WKB approximation. Implications of the supersymmetric structure of Pauli's Hamiltonian are also considered. In particular, the paramagnetisation of a non-interacting electron gas is related to a modified version of Witten's index. We also show that the supersymmetry in this system provides a new counterexample for the paramagnetic conjecture.

1. Introduction

For more than two decades the idea of supersymmetry (SUSY) has been inspiring physicists. Originally, SUSY has been introduced to formulate a quantum field theory which unifies bosons and fermions [1,2]. It is a fact that bosons and fermions are clearly distinguishable and SUSY is apparently broken in our present environment. Nevertheless, the SUSY idea has led to new insights in the studies of nuclear physics, condensed matter physics, statistical physics and mathematical physics [3]. In particular, supersymmetric quantum mechanics, which originally has been introduced in 1976 by Nicolai [4] and rediscovered in 1981 by Witten [5], nowadays attracts much attention. In this report we will present some of the recent developments in SUSY quantum mechanics.

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In Section 2 we will give the definition of SUSY in a quantum mechanical system. Section 3 deals with new developments in Witten's ($N = 2$)-realization of the SUSY algebra, which essentially is a pair of one-dimensional quantum systems. In particular, the SUSY-inspired semi-classical quantization condition in this realization, which differs from the usual WKB condition, has led to remarkable results. These are presented in the Sections 3.1 and 3.2. In Section 4 we will consider another realization of the SUSY algebra. This ($N = 1$)-realization leads to the Pauli Hamiltonian of an electron in an arbitrary magnetic field. We will discuss, in 4.1, the implications of this SUSY structure on the paramagnetism of a two-dimensional non-interacting electron gas. Section 4.2 deals with the three-dimensional case. Here we present a new counterexample to the paramagnetic conjecture.

2. Supersymmetric quantum mechanics

Following Nicolai [4] we call a quantum mechanical system characterized by a Hamiltonian H acting in some Hilbert space \mathcal{H} *supersymmetric* if there exist selfadjoint operators $Q_i = Q_i^\dagger$, $i = 1, 2, \dots, N$ called *supercharges*, which also act on states in \mathcal{H} and fulfill the following SUSY algebra:

$$\begin{aligned} \{Q_i, Q_j\} &:= Q_i Q_j + Q_j Q_i = H \delta_{ij}, \\ [Q_i, H] &:= Q_i H - H Q_i = 0, \quad i, j = 1, 2, \dots, N. \end{aligned} \quad (2.1)$$

In addition to this definition, it may be convenient to postulate the existence of a selfadjoint operator $P = P^\dagger$, called *Witten operator* or *Witten parity*, which anticommutes with the supercharges, and therefore commutes with the Hamiltonian, and whose square is equal to the identity [6-8]:

$$\{Q_i, P\} = 0, \quad [H, P] = 0, \quad P^2 = 1. \quad (2.2)$$

This operator allows to introduce the notion of "bosonic" and "fermionic" states independently of an underlying space-time symmetry. Actually, the Witten parity is sometimes also written in the form $P = (-1)^F$ where F is the fermion-number operator. Therefore, eigenstates of P with eigenvalue -1 are called "fermions" and those with eigenvalue +1 are said to be "bosons". It should be stressed, that these kinds of bosons and fermions may in general not be interpreted as particles with integer and half-integer spin, respectively. To make this point more clear let us introduce a bosonic subspace \mathcal{H}_B and a fermionic subspace \mathcal{H}_F :

$$\mathcal{H}_B := \{|\Psi\rangle \in \mathcal{H} | P|\Psi\rangle = +|\Psi\rangle\}, \quad \mathcal{H}_F := \{|\Psi\rangle \in \mathcal{H} | P|\Psi\rangle = -|\Psi\rangle\}. \quad (2.3)$$

Hence, any state $|\Psi\rangle \in \mathcal{H}$ can be decomposed into its bosonic and fermionic components as follows:

$$|\Psi\rangle = \begin{pmatrix} |\psi_B\rangle \\ |\psi_F\rangle \end{pmatrix} \tag{2.4}$$

The Hilbert space may be written as a product space $\mathcal{H} = \mathcal{H}_0 \otimes \mathbb{C}^2$. It is obvious, that the Witten operator is represented by the third Pauli matrix σ_3 :

$$P = 1 \otimes \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.5}$$

Therefore, it would be more appropriate to use the notion "spin-up" and "spin down" states (of a fictitious spin $-\frac{1}{2}$ -particle) instead of bosonic and fermionic states, respectively. Having in addition only cartesian degrees of freedom \mathcal{H}_0 is given by the space of square-integrable functions over the d -dimensional Euclidean space \mathbb{R}^d , $\mathcal{H}_0 = \mathcal{L}^2(\mathbb{R}^d)$, $d \in \mathbb{N}$. As P anticommutes with all supercharges the latter are necessarily of the form

$$\begin{aligned} Q_1 &:= \frac{i}{\sqrt{2}} \left(A_1^\dagger \otimes \sigma_+ - A_1 \otimes \sigma_- \right) = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & A_1^\dagger \\ -A_1 & 0 \end{pmatrix} \\ Q_2 &:= \frac{1}{\sqrt{2}} \left(A_2^\dagger \otimes \sigma_+ + A_2 \otimes \sigma_- \right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & A_2^\dagger \\ A_2 & 0 \end{pmatrix} \\ &\vdots \end{aligned} \tag{2.6}$$

where $A_i, i = 1, 2, \dots, N$; are operators acting on states in \mathcal{H}_0 and $\sigma_\pm := (\sigma_1 \pm i\sigma_2)/2$ are the usual raising and lowering operators for the eigenvalues of σ_3 . It is clear that the supercharges transform spin-up states (bosons) into spin-down states (fermions) and vice versa. This is the analogue of the well-known SUSY transformations in four-dimensional SUSY field theory.

The supersymmetry (2.1) has also implications on the spectral properties of the Hamiltonian H . First of all, we note that $H = 2Q_i^2 \geq 0$. That is, the Hamiltonian has only non-negative eigenvalues. Let us suppose that $|\Psi_r\rangle$ is an eigenstate of H with positive eigenvalue $E_r > 0$. Then it follows immediately from the algebra (2.1) that

$$|\bar{\Psi}_r\rangle = (2/E_r)^{1/2} Q_i |\Psi_r\rangle, \quad i = 1, 2, \dots, N, \tag{2.7}$$

is also an eigenstate with the same positive eigenvalue. Hence, all positive-energy eigenstates occur in spin-up (boson) spin-down (fermion) pairs. This, however, is in general not true for possible zero-energy eigenstates. If the groundstate energy of H is equal to zero, that is, there exists a state $|\Psi_0\rangle \in \mathcal{H}_0 \otimes \mathbb{C}^2$ such that

$$H|\Psi_0\rangle = 0, \tag{2.8}$$

then SUSY is said to be a good symmetry. This is, because (2.8) implies that the groundstate is invariant under SUSY transformations, $Q_i|\Psi_0\rangle = 0$. If the groundstate

energy of H is strictly positive then SUSY is said to be broken. Figure 1 shows typical spectra for good and broken SUSY, respectively.

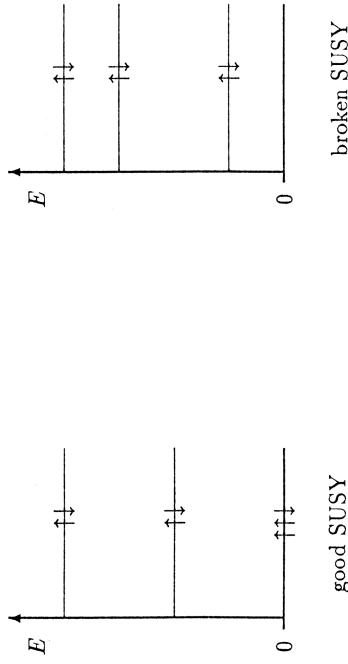


Figure 1. Typical spectra of the SUSY Hamiltonian H for good and broken SUSY. Positive energy eigenvalues always occur in spin-up-spin-down pair ($\uparrow\downarrow$).

In order to decide whether SUSY is a good symmetry Witten [6] introduced the following trace

$$\Delta(\beta) := \text{tr} (P e^{-\beta H}), \beta > 0. \tag{2.9}$$

This quantity is called *Witten index*. For a pure point spectrum of H this index is the difference of the number of spin-up states (\uparrow) and spin-down states (\downarrow) with zero energy:

$$\Delta(\beta) = N_{\uparrow}(E=0) - N_{\downarrow}(E=0). \tag{2.10}$$

Hence, $\Delta(\beta) \neq 0$ implies good SUSY. Obviously, the Witten index is independent of β . Indeed, the factor $e^{-\beta H}$ in (2.9) has only been introduced [9] for regularisation of the trace. The contributions of the positive-energy eigenstates cancel due to the pairwise degeneracy mentioned above. For a continuous spectrum this is not the case as the spectral densities for the spin-up and spin-down states are in general different. Here, the Witten index becomes β dependent [10].

3. Witten's realization

In this Section we will consider a particular ($N = 2$)-realization of the SUSY algebra (2.1) in the Hilbert space $\mathcal{H} = \mathcal{L}^2(\mathbb{R}) \otimes \mathbb{C}^2$. The two supercharges are given by (2.6) with

$$A_1 = A_2 \equiv A := \frac{i}{\sqrt{2m}} p + \phi(x). \tag{3.1}$$

Here p and x denote the usual momentum and position operators in $\mathcal{H}_0 = \mathcal{L}^2(\mathbb{R})$ and $\phi: \mathbb{R} \rightarrow \mathbb{R}$ is a piecewise continuously differentiable function called *SUSY potential*. The above realization has first been used by Witten [6] for studying the dynamical SUSY-breaking mechanism.

In terms of the SUSY potential ϕ the Hamiltonian reads ($\phi' := d\phi/dx$)

$$H = \left(\frac{p^2}{2m} + \phi^2(x) \right) \otimes 1 - \frac{\hbar}{\sqrt{2m}} \phi'(x) \otimes \sigma_3 \tag{3.2}$$

which describes a spin $-\frac{1}{2}$ point particle of mass $m > 0$ moving along the Euclidean line under the influence of the external scalar potential ϕ^2 and a magnetic field which is orthogonal to this axis and whose strength depends linearly on the slope of the SUSY potential. Obviously, the Hamiltonian is diagonal,

$$H = \begin{pmatrix} H_- & 0 \\ 0 & H_+ \end{pmatrix}, \tag{3.3}$$

and, therefore, the spin-up (bosonic) Hamiltonian H_- and spin-down (fermionic) Hamiltonian H_+ , which are defined as follows,

$$\begin{aligned} H_- : &= A^\dagger A = \frac{p^2}{2m} + \phi^2(x) - \frac{\hbar}{\sqrt{2m}} \phi'(x), \\ H_+ : &= AA^\dagger = \frac{p^2}{2m} + \phi^2(x) + \frac{\hbar}{\sqrt{2m}} \phi'(x), \end{aligned} \tag{3.4}$$

can be considered separately. This allows the alternative interpretation that Witten's realization (3.1) characterizes two non-interacting point particles of equal mass m moving along the real line under the influence of the external scalar potentials V_- and V_+ , respectively, where

$$V_\pm(x) := \phi^2(x) \pm \frac{\hbar}{\sqrt{2m}} \phi'(x). \tag{3.5}$$

Let us denote by $|\psi_r^+\rangle$ and $|\psi_r^-\rangle$ the eigenstates of H_+ and H_- , respectively, for the same positive eigenvalue $E_r > 0$:

$$H_\pm |\psi_r^\pm\rangle = E_r |\psi_r^\pm\rangle. \tag{3.6}$$

Then the SUSY transformation (2.7) implies the relations

$$|\psi_r^+\rangle = \frac{1}{\sqrt{E_r}} A |\psi_r^-\rangle, \quad |\psi_r^-\rangle = \frac{1}{\sqrt{E_r}} A^\dagger |\psi_r^+\rangle. \tag{3.7}$$

It should be noted that under the replacement $\phi \rightarrow -\phi$ the roles of the two Hamiltonians H_+ and H_- are interchanged, that is $H_\pm \rightarrow H_\mp$. Hence, the sign of

the SUSY potential may be fixed by some convention. The usual convention is that the groundstate energy belongs to a spin-up state (boson):

$$\inf \text{spec}(H) = \inf \text{spec}(H_-). \tag{3.8}$$

For good SUSY there exists in addition to the states (3.6) the ground state $|\psi_0\rangle$ defined in accordance with our convention (3.8) by

$$H_- |\psi_0\rangle = 0 \iff A |\psi_0\rangle = 0. \tag{3.9}$$

In the coordinate representation this state reads

$$\langle x | \psi_0 \rangle \equiv \psi_0(x) = \psi_0(0) \exp \left\{ -\frac{\sqrt{2m}}{\hbar} \int_0^x dz \phi(z) \right\} \tag{3.10}$$

which has to be square-integrable for SUSY to be a good symmetry. This requirement puts conditions on the SUSY potential ϕ . In Figure 2 we show typical ϕ 's which lead to good and broken SUSY, respectively. For continuous SUSY potentials the function ϕ must have an odd number of zeros (counted with their multiplicity) for SUSY to be good. A continuous SUSY potential with an even number of zeros necessarily leads to a broken SUSY as (3.10) will be not square-integrable. Consequently, if ϕ has a well-defined parity, an odd ϕ leads to good SUSY, whereas an even ϕ breaks SUSY [11].

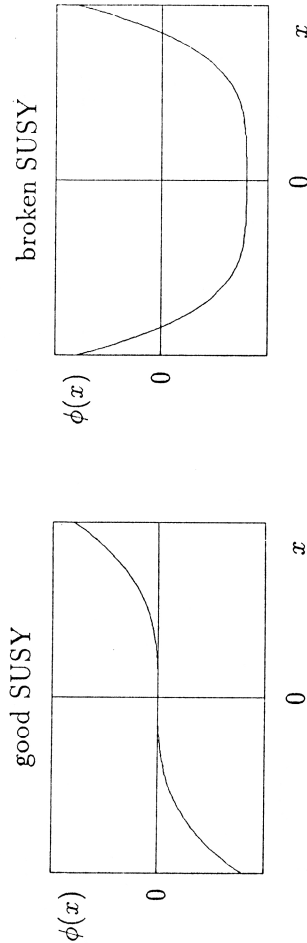


Figure 2. Typical SUSY potentials for good and broken SUSY

$$\begin{aligned} \phi(-x) &= -\phi(x) \longrightarrow V_\pm(-x) = V_\pm(x) \quad (\text{SUSY and parity are good}), \\ \phi(-x) &= \phi(x) \longrightarrow V_\pm(-x) \neq V_\pm(x) \quad (\text{SUSY and parity are broken}). \end{aligned} \tag{3.11}$$

The spectra of the two Hamiltonians (3.4) are related as follows:

$$\alpha(x) := \arcsin \left(\frac{\phi(x)}{\sqrt{E}} \right). \tag{3.17}$$

The left and right turning points $q_L \leq q_R$ are defined by $\phi^2(q_L) = E = \phi^2(q_R)$. We note that this quantization condition holds for both potentials V_{\pm} . This is not the case for the WKB formula which reads

$$\int_{x_L}^{x_R} dx \sqrt{2m[E - V_{\pm}(x)]} = \left(n + \frac{1}{2} \right) \pi \hbar, \quad n = 0, 1, 2, \dots \tag{3.18}$$

Here the turning points are given by $V_{\pm}(x_L) = E = V_{\pm}(x_R)$. Formally, the previous expression (3.16) can be derived from the WKB formula by inserting the explicit form (3.5) of V_{\pm} and treating ϕ to be independent of \hbar (eq.(3.15) clearly shows that this is in general not the case). An expansion in \hbar , see for example [15,16], then leads to the desired result (3.16).

There are two possible solutions for the roots of the turning-point- condition $\phi^2(q_L) = E = \phi^2(q_R)$ which can be related to good and broken SUSY. The first possibility $\phi(q_L) = -\phi(q_R) = \pm\sqrt{E}$ corresponds to good SUSY with the upper sign for H_+ and the lower one for H_- in accordance with the convention (3.8) (see also Figure 2). The second possibility, $\phi(q_L) = \phi(q_R) = \pm\sqrt{E}$, is related to a broken SUSY. More explicitly, formula (3.16) can be put into the form

$$\int_{q_L}^{q_R} dx \sqrt{2m[E - \phi^2(x)]} = (n + \nu)\pi\hbar, \quad n = 0, 1, 2, \dots, \tag{3.19}$$

where

$$\nu := \begin{cases} 0 & \text{for } H_- \text{ and good SUSY} \\ 1 & \text{for } H_+ \text{ and good SUSY} \\ \frac{1}{2} & \text{for } H_{\pm} \text{ and broken SUSY} \end{cases} \tag{3.20}$$

For good SUSY ($\nu = 0, 1$) this semi-classical quantization condition has first been suggested by Comtet, Bandrauk and Campbell [17] using the formal \hbar expansion mentioned above.

Let us note that this new quantization condition is in accordance with the spectral relations (3.12). This is generally not true for the WKB approximation. Furthermore, for good SUSY the condition (3.19) leads to the exact groundstate energy $E = 0$ of H_- . Thus being exact for the ground state and (as any semi-classical approximation) a good approximation for large quantum numbers n it may be conjectured that the SUSY approximation formula (3.19) might lead to estimates for energy eigenvalues with intermediate quantum numbers which are better than those of the WKB approximation (3.18). Indeed, for all so-called shape-invariant potentials [18] (the shape-invariance condition is identical to the factorizability of Schrödinger, Infeld and Hull [19]) it has

$$\begin{aligned} \text{spec}(H_-) \setminus \{0\} &= \text{spec}(H_+) && \text{(good SUSY)}, \\ \text{spec}(H_-) &= \text{spec}(H_+) && \text{(broken SUSY)}, \end{aligned} \tag{3.12}$$

Finally, let us mention that any one-dimensional standard problem, that is, the Schrödinger problem (in \mathcal{H}_0 for a point mass m in a scalar potential $U(x)$), can be put into this SUSY formalism. For example, knowing one (not necessarily square integrable) solution $\varphi(x)$ of the Schrödinger-like equation

$$-\frac{\hbar^2}{2m}\varphi''(x) + U(x)\varphi(x) = \varepsilon\varphi(x) \tag{3.13}$$

a family of solutions of the generalized Riccati equation [12]

$$\phi^2(x) - \frac{\hbar}{2m}\phi'(x) = U(x) - \varepsilon \tag{3.14}$$

is given by

$$\phi(x) := -\frac{\hbar}{\sqrt{2m}} \left(\frac{\varphi'(x)}{\varphi(x)} + \frac{\varphi^{-2}(x)}{\lambda + \int dx \varphi^{-2}(x)} \right). \tag{3.15}$$

It should be noted that not all parameters $\lambda \in \mathbb{R}$ will lead to an admissible SUSY potential ϕ such that H_{\pm} will be selfadjoint on \mathcal{H}_0 . However, for $\lambda \rightarrow \infty, \varepsilon$ being the groundstate energy of the Schrödinger problem (3.13) and $\varphi(x)$ being the corresponding normalized groundstate wavefunction, the SUSY potential (3.15) takes the form $\phi(x) = -(\hbar/\sqrt{2m})\varphi'(x)/\varphi(x)$ and will be admissible.

3.1. Semi-classical approximations

Recently, Witten's realization (3.1) of SUSY quantum mechanics has attracted much attention. This is mainly due to a new semi-classical approximation which leads for such SUSY systems to better results than the old WKB approximation does. In fact, for basically all exactly solvable problems the new approximation leads to exact energy spectra. This is in contrast to the WKB approach which in general requires ad hoc Langer-like modifications for those systems in order to yield the exact spectrum. The new approximation formula, which can be derived from Feynman's path integral using a modified stationary-action-principle [11,13,14], reads

$$\int_{q_L}^{q_R} dx \sqrt{2m[E - \phi^2(x)]} = \left(n + \frac{1}{2} - \frac{a(q_L) - a(q_R)}{2\pi} \right) \pi \hbar, \quad n = 0, 1, 2, \dots, \tag{3.16}$$

where

been found that the SUSY formula (3.19) does provide the exact spectrum. The shape-invariant potentials can be put into three classes [18]:

Class 1:

$$\begin{aligned} \phi_1(x) &= \frac{\hbar}{\sqrt{2m}}(af_1(x) + b) \quad \text{with} \quad f_1'(x) = pf_1^2(x) + qf_1(x) + r, \\ V_{\pm}^{(1)}(x) &= \frac{\hbar^2}{2m} [a(a \pm p)f_1^2(x)a(2b \pm q)f_1(x) + (b^2 \pm ar)]. \end{aligned} \quad (3.21)$$

Class 2:

$$\begin{aligned} \phi_2(x) &= \frac{\hbar}{\sqrt{2m}}(af_2(x) + b/f_2(x)) \quad \text{with} \quad f_2'(x) = pf_2^2(x) + q, \\ V_{\pm}^{(2)}(x) &= \frac{\hbar^2}{2m} [a(a \pm p)f_2^2(x) + b(b \pm q)/f_2^2(x) + 2ab \pm (aq - bp)]. \end{aligned} \quad (3.22)$$

Class 3:

$$\begin{aligned} \phi_3(x) &= \frac{\hbar}{\sqrt{2m}} \left(a + b\sqrt{pf_3^2(x) + q} \right) / f_3(x) \quad \text{with} \quad f_3'(x) = \sqrt{pf_3^2(x) + q}, \\ V_{\pm}^{(3)}(x) &= \frac{\hbar^2}{2m} \left[\frac{a^2 + pq(b \pm 1)}{f_3^2(x)} + \frac{\sqrt{pf_3^2(x) + q}}{f_3^2(x)} a(2b \pm 1) + b^2 p \right]. \end{aligned} \quad (3.23)$$

Here $a, b, p, q, r \in \mathbb{R}$ are arbitrary potential parameters. Shape invariance means that the potential $V_{\pm}^{(i)}$ can be written in terms of $V_{\pm}^{(i)}$ after appropriate redefinitions of the potential parameters. For example, in classes 1 and 2 we may perform the reparameterization $(p, q, r) \rightarrow (-p, -q, -r)$ which results in $V_{\pm}^{(1,2)} \leftrightarrow V_{\mp}^{(1,2)}$. We note that the potentials belonging to class 2 possess additional reparameterisation invariances:

$$\begin{aligned} a &\rightarrow -(a \pm p) \implies \phi_2(x) \rightarrow \phi_2(x) \rightarrow -(a \pm p)f_2(x) + b/f_2(x) \\ b &\rightarrow -(b \mp q) \implies \phi_2(x) \rightarrow a f_2(x) - (b \mp q)/f_2(x). \end{aligned} \quad (3.24)$$

In both cases the full potentials are only shifted by an additional constant, $V_{\pm}^{(2)}(x) \rightarrow V_{\pm}^{(2)}(x) + \text{const}$. The particular form of the SUSY potential ϕ_2 allows one to choose parameters such that SUSY will either be good or broken. These facts have first been pointed out by Suparmi [20] (see also [16,21]). Let us also mention that the shape-invariance condition, which is equivalent to the factorization condition, implies also dynamical symmetries in addition to SUSY. For a recent discussion of the embedding

of this SUSY structure into the dynamical group structure of shape-invariant potentials we refer to the work by Barut and Roy [22].

As already pointed out, the interesting observation is that the new approximation formula (3.19) leads to the exact spectrum of all those shape-invariant potentials. Whereas the classes 1 and 3 allow only for a good SUSY, class 2 also provides broken SUSY examples [13,16]. Let us consider the example of the radial harmonic oscillator which belongs to class 2 with $f_2(x) := x, a := m\omega/\hbar > 0, q := 1$ and $p := 0$. Here $\mathcal{H}_0 = \mathcal{L}^2(\mathbb{R}^+)$ with Dirichlet boundary condition at $x = 0$. The choice $b := -l - l \leq -1$ leads to the SUSY potential

$$\phi(x) = \sqrt{\frac{m}{2}} \omega x - \frac{(l+1)\hbar}{\sqrt{2mx}} \quad (3.26)$$

with good SUSY. The corresponding full potentials read

$$\begin{aligned} V_{-}^{(2)}(x) &= \frac{m}{2} \omega^2 x^2 + \frac{l(l+1)\hbar^2}{2mx^2} - \hbar\omega \left(l + \frac{3}{2} \right), \\ V_{+}^{(2)}(x) &= \frac{m}{2} \omega^2 x^2 + \frac{(l+1)(l+2)\hbar^2}{2mx^2} - \hbar\omega \left(l + \frac{1}{2} \right). \end{aligned} \quad (3.27)$$

Application of the semi-classical quantization condition (3.19) gives [20]

$$E_n = 2\hbar\omega(n + \nu). \quad (3.28)$$

This is indeed the exact spectrum for (3.27). For the alternative choice $b = l \geq 0$ we have

$$\phi(x) = \sqrt{\frac{m}{2}} \omega x + \frac{\hbar}{\sqrt{2mx}} \quad (3.29)$$

which gives a broken SUSY. The full potentials read in this case

$$\begin{aligned} V_{-}^{(2)}(x) &= \frac{m}{2} \omega^2 x^2 + \frac{l(l+1)\hbar^2}{2mx^2} + \hbar\omega \left(l - \frac{1}{2} \right), \\ V_{+}^{(2)}(x) &= \frac{m}{2} \omega^2 x^2 + \frac{l(l-1)\hbar^2}{2mx^2} + \hbar\omega \left(l + \frac{1}{2} \right). \end{aligned} \quad (3.30)$$

Here, the formula (3.19) has to be applied with $\nu = 1/2$. Again the exact spectrum

$$E_n = \hbar\omega(2n + 2l + 1) \quad (3.31)$$

can be obtained [20]. We finally note that the WKB approximation (3.18) provides these exact results only after making the Langer modification $l(l+1) \rightarrow (l+1/2)^2$.

3.2. Numerical results for a class of power potentials

Being superior to the WKB approximation in the case of shape-invariant problems, there naturally arises the question whether the new approximation (3.19) also provides better estimates for not exactly solvable systems. To answer this questions we have investigated a class of power SUSY potentials of the form [14]

$$\phi(x) := \frac{\hbar a}{\sqrt{2m}} x^d \quad \text{for } x \geq 0. \tag{3.32}$$

Here $a > 0$ and $d = 1, 2, \dots$, are free parameters. Note that the above definition is only for the positive Euclidean half-line. For $x < 0$ we may define the SUSY potential either through an antisymmetric or symmetric continuation which leads to a good and broken SUSY, respectively:

$$\begin{aligned} \phi(-x) &:= -\phi(x) \quad \text{for good SUSY,} \\ \phi(-x) &:= \phi(x) \quad \text{for broken SUSY.} \end{aligned} \tag{3.33}$$

The full potentials read

$$\begin{aligned} V_{\pm}(x) &= \frac{\hbar^2}{2m} (a^2 x^{2d} \pm a d |x|^{d-1}) = V_{\pm}(-x) \quad \text{for good SUSY,} \\ V_{\pm}(x) &= \frac{\hbar^2}{2m} (a^2 x^{2d} \pm a d x |x|^{d-2}) = V_{\mp}(-x) \quad \text{for broken SUSY,} \end{aligned} \tag{3.34}$$

For the SUSY potential (3.32) the new semi-classical formula (3.19) can be evaluated analytically with the result

$$E_n = \frac{\hbar^2 a^2}{2m} \left[\frac{\Gamma(\frac{3d+1}{2d}) \sqrt{\pi}}{\Gamma(\frac{2d+1}{2d})} (n + \nu) \right]^{2d/(d+1)} \tag{3.35}$$

These energy eigenvalues are in general not exact. For arbitrary d only the case $n = \nu = 0$ in (3.35) gives the exact result, which is the groundstate energy for good SUSY. The associated wavefunction (3.10) reads explicitly

$$\psi_0(x) = \psi_0(0) \exp \left\{ -\frac{a}{d+1} |x|^{d+1} \right\}. \tag{3.36}$$

Another case where (3.35) becomes exact is the limit $d \rightarrow \infty$:

$$\lim_{d \rightarrow \infty} E_n = \frac{\hbar^2 \pi^2}{8m} (n + \nu)^2. \tag{3.37}$$

That this is indeed the exact spectrum can be seen by realizing that (3.34) become infinite square-well potentials:

$$\lim_{d \rightarrow \infty} V_{\pm}(x) = \begin{cases} 0 & \text{for } |x| < 1 \\ \infty & \text{for } |x| > 1 \end{cases} \tag{3.38}$$

This limit has to be taken with some care. Actually, the Hilbert space changes in this limit, too: $\mathcal{L}^2(\mathbb{R}) \rightarrow \mathcal{L}^2([-1, 1])$. Hence, now one has to specify boundary conditions at $x = \pm 1$ in order to have a well-defined problem. The type of boundary conditions which has to be chosen is related to the requirement that the SUSY structure, that is, good or broken SUSY, is conserved after performing the limit $d \rightarrow \infty$. It is obvious that for this one has to impose, for good SUSY, Neumann conditions at $x = \pm 1$ for V_- and Dirichlet condition at $x = \pm 1$ for V_+ . Thus parity as well as SUSY remain to be good symmetries. For broken SUSY and broken parity we have to choose for V_- Dirichlet conditions at $x = -1$ and Neumann conditions at $x = 1$, and for V_+ vice versa. Let us note that the spectrum (3.35) also appears in the contribution of Uzes, Barut and Kapuscik [23] to these proceedings, where they discuss the quantization of a constrained complex non-linear oscillator. Indeed, the Hamiltonian (3.2) in Witten's realization can be derive from a more general SUSY Hamiltonian with second-class constraints [24].

For finite d we have compared the eigenvalues (3.35) and those obtained via the WKB formula (3.18) with the numerically exact eigenvalues of the Schrödinger equation for the full potentials (3.34). In Figure 3 and 4 we plot the relative deviations (in%)

$$\Delta := (E_{exact} - E_{approx})/E_{exact} \tag{3.39}$$

for the cases $d = 2, 3$ and for good and broken SUSY, respectively. For the numerical evaluation we used units such that $m = \hbar = a = 1$. As can clearly be seen in these Figures, the new approximation formula provides estimates for the eigenvalues which are above of those of the WKB approximation. Except for the first few eigenvalues for good SUSY we made the observation that the new formula (3.19) always overestimates the exact energy value whereas the WKB formula (3.18) gives underestimations. We have made this observation not only for the class of power potentials characterized by (3.32) but for all systems investigated so far. In Figure 5 we present results for $\phi(x) = \sinh(x)$ and $\phi(x) = \cosh(x)$ with good and broken SUSY, respectively, showing the same tendency [14].

$B(x, y) := \partial_x a_2(x, y) - \partial_y a_1(x, y) - \partial_y a_1(x, y) - \partial_x a_1(x, y)$ which is perpendicular to the $x - y$ -plane
The corresponding SUSY Hamiltonian $H = 2Q^2$ explicitly reads

$$H = \frac{1}{2m} \sum_{i=1}^2 \left(p_i - \frac{e}{c} a_i(x, y) \right)^2 \otimes 1 - \mu_B B(x, y) \otimes \sigma_3 \quad (4.2)$$

where $\mu_B := e\hbar/(2mc)$ is the Bohr magneton. Hence, the Hamiltonian (4.2) is identical with the Pauli Hamiltonian for a two-dimensional electron of charge e , mass m and gyromagnetic ratio equal to two. This realization of the SUSY algebra (2.1) has first been given by de Crombrugge and Rittenberg [25]. However, the observation that the Pauli Hamiltonian factorizes has already been made by Kramers [26]. In this realization the bosonic and fermionic states are the usual spin-up and spin-down states of the electron. Let us also note that there is a close relation between the supercharge (4.1) and the Dirac operator. For more details we refer to the excellent book by Thaller [8] which makes extensive use of SUSY in the discussion of the Dirac equation.

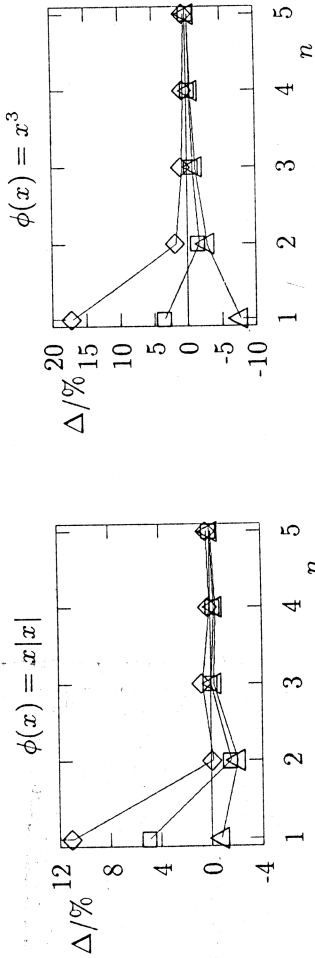


Figure 3. Relative deviations (3.39) for good SUSY. The results (3.35) obtained from the new formula (3.19) are indicated by the symbol \diamond . Those obtained from the WKB formula (3.18) are shown by \square for V_- and by \triangle for V_+ . The lines are drawn to guide the eye.

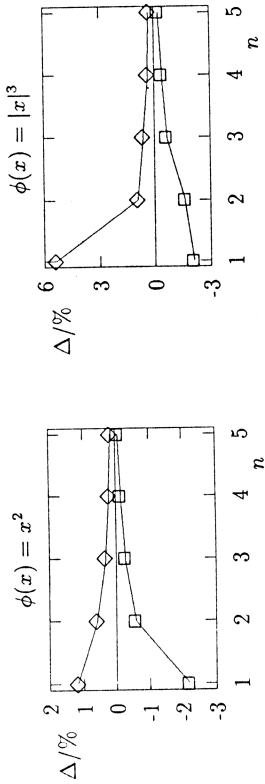


Figure 4. Same as Figure 3 but for broken SUSY. Again the results (3.25) are shown by \diamond . Here the WKB formula (3.18) gives identical results (\square) for V_+ and for V_- due to the symmetry (3.34).

We conclude that the new approximation (3.19) in general does not provide estimates better than those obtained from the WKB approximation (3.18). However, combining both by taking the mean value will give much better estimates [14].

4. Supersymmetry in the Pauli Hamiltonian

In this Section we will consider a $(N = 1)$ -realization of the algebra (2.1) in the Hilbert space $\mathcal{H} = \mathcal{L}^2(\mathbb{R}^2) \otimes \mathbb{C}^2$. Here the supercharge $Q \equiv Q_1$ is defined by

$$Q := \frac{1}{\sqrt{4m}} \sum_{i=1}^2 \left[\left(p_i - \frac{e}{c} a_i(x, y) \right) \otimes \sigma_i \right] \quad (4.1)$$

where e denotes the charge of a point mass m , c is the speed of light, and $a_i : \mathbb{R}^2 \rightarrow \mathbb{R}$ are components of an external vector potential characterizing a magnetic field of strength

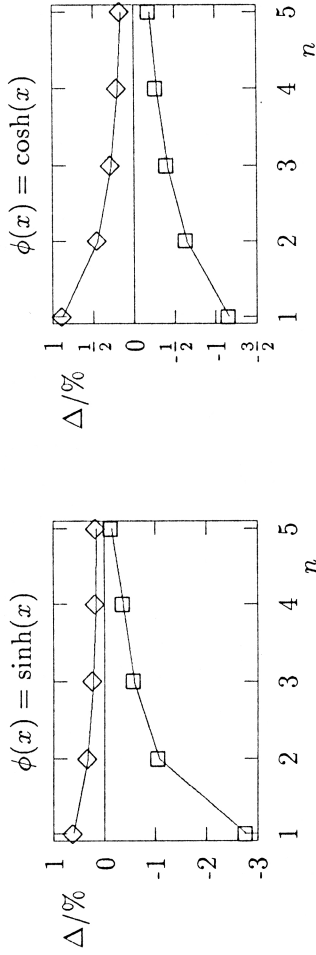


Figure 5. Same as Figure 3 for the SUSY potential $\phi(x) = \sin h(x)$ (good SUSY) and $\phi(x) = \cos h(x)$ (broken SUSY). The symbol \diamond indicates results from the new approximation, the symbol \square those from the WKB approximation. In both cases the WKB formula leads for V_+ to the same values as for V_- .

Due to a famous result of Aharonov and Casher [27] (see also [7,8]) the ground-state energy of the Hamiltonian (4.2) is zero and belongs either to spin-up or spin-down states. Therefore, SUSY is a good symmetry for an arbitrary magnetic field strength B . The nature of the groundstates can, as before, be fixed by some convention which essentially corresponds to an appropriate choice of the overall sign of B . By choosing this sign such that the net magnetic flux through the $x - y$ plane is positive the groundstate energy belongs to spin-up states only. The degeneracy of the groundstate is given by

$$g := \left[\left[\frac{1}{\Phi_0} \int_{\mathbb{R}^2} dx dy B(x, y) \right] \right] \quad (4.3)$$

where $\Phi_0 := 2\pi\hbar c/e$ is the flux quantum and $[z]$ denotes the largest integer which is strictly less than z . For a pure point spectrum (this occurs for example in the case of a constant magnetic field [28]) the degeneracy g is identical to the Witten index (2.9):

$$g = \Delta(\beta) = N_+(E = 0). \tag{4.4}$$

The equality of the two right-hand-sides of (4.3) and (4.4) is a physical realization of the Atiyah-Singer index theorem relating the algebraic index of an operator with its topological index.

4.1. Paramagnetism of the two-dimensional electron gas

The SUSY of the Pauli Hamiltonian (4.2) allows to study its paramagnetic properties for arbitrary magnetic fields B . Let us first assume that the magnetic field is such that (4.2) has a pure point spectrum. Then, because of SUSY, we know that all subspaces with a strictly positive energy eigenvalue contain pairs of spin-up and spin-down states. Hence, the paramagnetic contribution M_p to the total magnetisation originates from the zero-energy states only, each of which contributes one Bohr magneton: $M_p = g\mu_B$. We note that this magnetisation can be written in terms of the Witten operator:

$$M_p = \mu_B \Delta(\beta). \tag{4.5}$$

This relation, however, is no longer true if the spectrum of (4.2) contains also a continuous part starting at ε_c , the so-called mobility edge. That is, we assume a discrete spectrum with eigenvalues below ε_c and a continuous spectrum given by $[\varepsilon_c, \infty)$. Let us introduce another regularisation for the trace of the Witten operator $P = \sigma_3$ leading to a modified Witten index [29]:

$$\tilde{\Delta}(\varepsilon) := \text{tr}(P\Theta(H - \varepsilon)), \quad \varepsilon > 0. \tag{4.6}$$

Here Θ denotes the unit-step function taking the value zero for negative arguments and the value one for positive arguments. It is obvious that (4.6) agrees with the original Witten index if H has a pure point spectrum. However, in the case of a finite mobility edge $\varepsilon_c < \infty$ we have:

$$\tilde{\Delta}(\varepsilon) = g + \Theta(\varepsilon - \varepsilon_c) \int_{\varepsilon_c}^{\varepsilon} dE (\rho_1(E) - \rho_2(E)), \tag{4.7}$$

where ρ_1 and ρ_2 denote the densities of the spin-up and spin-down energy eigenstates, respectively. The definition (4.6) allows to express the magnetisation in terms of this modified Witten index:

$$M_p = \mu_B \tilde{\Delta}(\varepsilon_F) \tag{4.8}$$

where ε_F denotes the Fermi energy of the two-dimensional non-interacting electron gas in the magnetic field B . Also (4.8) is only exact for absolute zero temperature $T = 0$, it will be good for $k_B T \ll \varepsilon_F$ (k_B : Boltzmann's constant). Otherwise, the Fermi energy in (4.8) will have to be replaced by the temperature-dependent chemical potential.

This result suggests that the existence and position ε_c of a mobility edge for a certain magnetic field configuration may be verified experimentally as nowadays the Fermi energy in two-dimensional semiconductors can be chosen rather freely in the preparation of the samples.

4.2. The three-dimensional electron and the paramagnetic conjecture

The above results on the two-dimensional electron can more or less be taken over to the three-dimensional case. Here the supercharge may be defined as follows [25]:

$$Q' := \frac{1}{\sqrt{4m}} \sum_{i=1}^3 \left[\left(p_i - \frac{e}{c} a_i(\vec{r}) \right) \otimes \sigma_i \right], \tag{4.9}$$

where $\vec{r} := (x, y, z)$ and $\vec{p} := (p_1, p_2, p_3)$ are the position and momentum operators on $\mathcal{H}_0 = \mathcal{L}^2(\mathbb{R}^3)$, respectively. The corresponding Hamiltonian $H' := 2Q'^2$ becomes the Pauli Hamiltonian of the three-dimensional electron. Let us note that the Witten operator $P = \sigma_3$ does, in general, not commute with H' . That is, the relations (2.2) are no longer true. Only in the special case where the magnetic field $\vec{B}(\vec{r}) := \vec{\nabla} \times \vec{a}(\vec{r}) = B(x, y)\vec{e}_z$ is translational invariant along one axis, say the z -axis, all the results found above remain valid. This is because of the relation $H' = H + p_3^2/2m$. Hence, any subspace corresponding to a fixed eigenvalue of p_3 constitutes a two-dimensional SUSY system as discussed in Section 4.1.

Nevertheless, the relation (4.8) between the magnetisation and the modified Witten index is still valid for arbitrary magnetic fields. However, the Aharonov-Casher result cannot be taken over to three dimension. Indeed, here it may be possible that SUSY is broken. For a pure point spectrum of H' this has the surprising consequence of a vanishing magnetisation as the Witten index and its modified version is zero in this case.

Another consequence of the SUSY is that it provides a counterexample to the paramagnetic conjecture due to Høegreve, Schrader and Seiler [30]. This conjecture states that the groundstate energy of the general Pauli Hamiltonian

$$H_P(\vec{a}, V) := \frac{1}{2m} \left(\vec{p} - \frac{e}{c} \vec{a} \right)^2 - \mu_B (\vec{\nabla} \times \vec{a}) \cdot \vec{\sigma} + V \tag{4.10}$$

is always less or equal to that with zero magnetic field:

$$\inf \text{spec}(H_P(\vec{a}, V)) \leq \inf \text{spec}(H_P(0, V)). \tag{4.11}$$

A proof of this inequality exist for arbitrary scalar potentials V and magnetic fields of the form $\vec{B}(\vec{r}) = B(x, y)\vec{e}_z$ [31]. However, in the general case Avron and Simon [32]

found a counter- example. Nevertheless, it is believed, see ref. [7] p.131, that (4.11) "... still holds for general \vec{a} and selected sets of V ...". Here we note that for the particular set $\{V=0\}$ the factorizability of $H_P(\vec{a}, 0) \equiv H' = 2Q'^2 \geq 0$ implies the inequality

$$\inf \text{spec}(H_P(\vec{a}, 0)) \geq \inf \text{spec}(H_P(0, 0)) = 0. \quad (4.12)$$

This inequality is in the opposite direction of the paramagnetic conjecture (4.11). For arbitrary magnetic fields such that SUSY is a good symmetry the inequality (4.12) can be replaced by an equality. However, for any magnetic field which does break SUSY we have a strict inequality in (4. 12) and thereby a counterexample to the conjecture (4.11). Or vice versa, for any magnetic field, for which the paramagnetic conjecture with $V = 0$ can be proven, SUSY will be a good symmetry and hence equality holds in (4. 12).

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latter case one may start with a finite configuration space such as a d - dimensional box of volume L^d , $L > 0$, and, after the evaluation of the trace (2.9), take (if possible) the limit $L \rightarrow \infty$ in an appropriate way.

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